

On a class fractional Schrödinger equations with indefinite potential involving critical exponential growth *

Manassés de Souza[†] and Yane Lisley Araújo

Departamento de Matemática
Universidade Federal da Paraíba
58051-900 João Pessoa, PB, Brazil

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Abstract

It is established the existence and multiplicity of weak solutions for a class of nonlocal equations involving the fractional laplacian, nonlinearities with critical exponential growth and potentials this is which may change sign. The proofs of our existence results rely on minimization methods in combination with the mountain-pass theorem.

Keywords: Variational methods; critical points; Trudinger-Moser inequality; fractional laplacian.

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1 Introduction

In this paper we investigate the existence and multiplicity of weak solutions for the following class of equations

$$(-\Delta)^{1/2}u + V(x)u = f(x, u) + h \quad \text{in } \mathbb{R}, \quad (1.1)$$

where $V : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous potential which may change sign, the nonlinearity $f(x, s)$ behaves like $\exp(\alpha_0 s^2)$ when $|s| \rightarrow +\infty$ for some $\alpha_0 > 0$, h belongs to the dual of an appropriated functional space and $(-\Delta)^{1/2}$ is the fractional laplacian. The fractional laplacian $(-\Delta)^{1/2}$ of a measurable function $u : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$(-\Delta)^{1/2}u(x) = -\frac{1}{2\pi} \int_{\mathbb{R}} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^2} dy. \quad (1.2)$$

Recently, a great attention has been focused on the study of fractional and non-local operators of elliptic type, both for the pure mathematical research and in view of concrete real-world

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[†]Corresponding author, E-mail address: manassesxavier@hotmail.com

applications. This type of operators arises in a quite natural way in many different contexts, such as, among the others, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves. The literature about non-local operators and on their applications is, therefore, very interesting and, up to now, quite large (see, for instance, [5] for an elementary introduction to this topic and for a still not exhaustive list of related references).

In order to study variationally (1.1) we will consider a suitable subspace of the fractional Sobolev space $H^{1/2}(\mathbb{R})$. We recall that $H^{1/2}(\mathbb{R})$ is defined as

$$H^{1/2}(\mathbb{R}) := \left\{ u \in L^2(\mathbb{R}) : \frac{|u(x) - u(y)|}{|x - y|} \in L^2(\mathbb{R} \times \mathbb{R}) \right\};$$

endowed with the norm

$$\|u\|_{1/2,2} := \left(\int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^2} dx dy + \int_{\mathbb{R}} |u|^2 dx \right)^{1/2}.$$

The term

$$[u]_{1/2,2} := \left(\int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^2} dx dy \right)^{1/2}$$

is the so-called *Gagliardo semi-norm* of function u .

We assume suitable conditions on the potential $V(x)$ with which we will be able to consider a variational framework based in the space X given by

$$X = \left\{ u \in H^{1/2}(\mathbb{R}) : \int_{\mathbb{R}} V(x) u^2 dx < \infty \right\}.$$

More precisely, we assume throughout this paper the following assumptions on $V(x)$:

(V_1) There exists a positive constant B such that

$$V(x) \geq -B \text{ for all } x \in \mathbb{R};$$

(V_2) The infimum

$$\lambda_1 := \inf_{\substack{u \in X \\ \|u\|_2 = 1}} \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x) - u(y))^2}{|x - y|^2} dx dy + \int_{\mathbb{R}} V(x) u^2 dx \right)$$

is positive;

(V_3) $\lim_{R \rightarrow \infty} \nu(\mathbb{R} \setminus \overline{B}_R) = +\infty$, where

$$\nu(G) = \begin{cases} \inf_{\substack{u \in X_0(G) \\ \|u\|_2 = 1}} \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x) - u(y))^2}{|x - y|^2} dx dy + \int_G V(x) u^2 dx & \text{if } G \neq \emptyset; \\ \infty & \text{if } G = \emptyset. \end{cases}$$

Here G is a open set in \mathbb{R} , $X_0(G) = \{u \in X : u = 0 \text{ in } \mathbb{R} \setminus G\}$ and \overline{B}_R is the closed ball with center at origin and radius R .

The hypotheses $(V_1) - (V_2)$ will ensure that the space X is Hilbert when endowed with the inner product

$$\langle u, v \rangle = \frac{1}{2\pi} \left(\int_{\mathbb{R}^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^2} dx dy \right) + \int_{\mathbb{R}} V(x)uv dx, \quad u, v \in X,$$

to which corresponds the norm $\|u\| = \langle u, u \rangle^{1/2}$ (cf. Section 2).

In this context, we assume that $h \in X^*$ (dual space of X) and say that $u \in X$ is a weak solution for the problem (1.1) if the following equality holds:

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^2} dx dy + \int_{\mathbb{R}} V(x)uv dx = \int_{\mathbb{R}} f(x, u)v dx + (h, v), \quad (1.3)$$

for all $v \in X$, where (\cdot, \cdot) denotes the duality pairing between X and X^* .

When a weak solution u has sufficient regularity, it is possible to have a pointwise expression of the fractional laplacian as (1.2). See [20], for example.

As we already mentioned, we are interested in the case that the nonlinearity $f(x, s)$ has the maximal growth which allows us to treat problem (1.1) variationally in X . Precisely, we will assume sufficient conditions so that a weak solution of (1.1) turn out to be critical points of the Euler functional $I : X \rightarrow \mathbb{R}$ defined by

$$I(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}} F(x, u) dx - (h, u), \quad \text{where } F(x, s) = \int_0^s f(x, t) dt.$$

In order to better describe the hypotheses on $f(x, s)$ we recall some well known facts about the limiting Sobolev embedding theorem in 1-dimension. If $s \in (0, 1/2)$, Sobolev embedding states that $H^s(\mathbb{R}) \hookrightarrow L^{2_s^*}(\mathbb{R})$, where $2_s^* = 2/(1 - 2s)$, for this case, the maximal growth of the nonlinearity $f(x, s)$ which allows to treat problem (1.1) variationally in $H^s(\mathbb{R})$ is given by $|s|^{2_s^*}$ as $|s| \rightarrow +\infty$. If $s = 1/2$, Sobolev embedding states that $H^{1/2}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ for any $q \in [2, +\infty)$, but $H^{1/2}(\mathbb{R})$ is not continuous embedded in $L^\infty(\mathbb{R})$, for details see [5, 13]. In the borderline case $s = 1/2$, the maximal growth which allows to treat problem (1.1) variationally in $H^{1/2}(\mathbb{R})$ is motivated by Trudinger-Moser inequality proved by H. Kozono, T. Sato and H. Wadade [9] and T. Ozawa [13]. Precisely, they proved that there exist positive constants ω and C such that for all $u \in H^{1/2}(\mathbb{R})$ with $\|(-\Delta)^{1/4}u\|_2 \leq 1$,

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1) dx \leq C \|u\|_2^2, \quad \forall \alpha \in (0, \omega]. \quad (1.4)$$

See also the pioneering works [12, 18]. Motivated by (1.4) we say that $f(x, s)$ has *critical exponential growth* when for all $x \in \mathbb{R}$, there exists $\alpha_0 > 0$ such that

$$\lim_{|s| \rightarrow +\infty} f(x, s) e^{-\alpha |s|^2} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0. \end{cases}$$

Now we state our main assumptions for the nonlinearity $f(x, s)$. In order to find weak solutions (1.1) through variational methods we will assume the following general hypotheses:

(f₁) $0 \leq \lim_{s \rightarrow 0} \frac{f(x, s)}{s} < \lambda_1$, uniformly in x ;

(f₂) $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, it has critical exponential growth and is locally bounded in s , that is, for any bounded interval $J \subset \mathbb{R}$, there exists $C > 0$ such that $|f(x, s)| \leq C$ for every $(x, s) \in \mathbb{R} \times J$;

(f₃) there exists $\theta > 2$ such that

$$0 < \theta F(x, s) := \theta \int_0^s f(x, t) dt \leq s f(x, s), \quad \text{for all } (x, s) \in \mathbb{R} \times \mathbb{R} \setminus \{0\};$$

(f₄) there exist constants $s_0, M_0 > 0$ such that

$$0 < F(x, s) \leq M_0 |f(x, s)|, \quad \text{for all } |s| \geq s_0 \quad \text{and } x \in \mathbb{R};$$

(f₅) there exist constants $p > 2$ and C_p such that, for all $s \geq 0$ and $x \in \mathbb{R}$,

$$f(x, s) \geq C_p s^{p-1},$$

with $C_p > \left[\frac{\alpha_0(p-2)}{2\pi\kappa\omega p} \right]^{(p-2)/2} S_p^p$, where S_p is given in (2.3).

We point out that the hypotheses (f₁) – (f₅) have been used in many papers to find a variational solution using the classical Mountain Pass Theorem introduced by Ambrosetti and Rabinowitz in the celebrated paper [1], see for instance [6, 7, 8]. A simple model of a function that verifies our assumptions is $f(x, s) = C_p |s|^{p-2} s + 2s(e^{s^2} - 1)$ for $(x, s) \in \mathbb{R} \times \mathbb{R}$.

We next state our main results.

Theorem 1.1. *Suppose that (V₁) – (V₃) and (f₁) – (f₅) are satisfied. Then there exists $\delta_1 > 0$ such that for each $0 < \|h\|_* < \delta_1$, problem (1.1) has at least two weak solutions. One of them with positive energy, while the other one with negative energy.*

Theorem 1.2. *Under the same hypotheses in Theorem 1.1, the problem without the perturbation, that is $h \equiv 0$, has a nontrivial weak solution with positive energy.*

Remark 1.3. Our work was motivated by Iannizzotto and Squassina [8] and some papers that have appeared in the recent years concerning the study of (1.1) by using purely variational approach, see [3, 15, 16] and references therein. Our goal is to extend and complement the results in [3, 8, 15, 16] in sense that we consider critical exponential growth on the nonlinearity and a class of potentials $V(x)$ which may change sign, vanish and be unbounded.

Remark 1.4. By examining the literature, we notice that many authors, considering different ways, have established the existence of solutions for problems involving the standard laplacian

$$-\Delta u + V(x)u = g(x, u), \quad x \in \mathbb{R}^N, \quad (1.5)$$

see e.g. [1, 2] for the case where $g(x, s)$ has subcritical growth in the Sobolev sense, and see e.g. [6, 7, 10, 19] for the case where $g(x, s)$ has critical growth in the Trudinger-Moser sense. The existence of solutions has been discussed under various conditions on the potential $V(x)$. It is worthwhile to remark that in these works different hypotheses are assumed on $V(x)$ in order to overcome the problem of “lack of compactness”, typical of elliptic problems defined in unbounded domains and involving nonlinearities in critical growth range. Specifically, in [2, 14] it is assumed that the potential is continuous and uniformly positive. Furthermore, it is assumed one of the following conditions:

- (a) $V(x) \nearrow +\infty$ as $|x| \rightarrow +\infty$;
- (b) For any $A > 0$, the level set $\{x \in \mathbb{R}^N : V(x) \leq A\}$ has finite Lebesgue measure.

Each of these conditions guarantee that the space

$$E := \left\{ u \in W^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 \, dx < \infty \right\}$$

is compactly embedded in the Lebesgue space $L^q(\mathbb{R}^N)$ for all $q \geq N$.

We point out (V_3) generalizes these two conditions. In special, it should be mentioned that the conditions $(V_1) - (V_3)$ were already considered by B. Sirakov [17] to study (1.5) when $g(x, u)$ has subcritical growth in the Sobolev sense.

Remark 1.5. Similar to [4, 6, 7, 8] we will use minimization to find the first solution with negative energy, and the Mountain Pass Theorem to guarantee the existence of the second solution with positive energy. First we need to check some conditions concerning the mountain pass geometry and the compactness of the associated-Euler functional. In our argument, it is crucial a version of the Trudinger-Moser inequality to space the X and a version of a Concentration-Compactness Principle due to P. -L. Lions [11] to the space X (cf. Section 2). Our main difficulties are the involved operator which is nonlocal and critical exponential growth on the nonlinearity.

The outline of the paper is as follows: Section 2 contains some preliminary results. Section 3 contains the variational framework and we also check the geometric conditions of the associated functional. Section 4 deals with Palais-Smale condition and Section 5 treats with the minimax level. Finally in Section 6, we complete the proofs of our main results.

Hereafter, C, C_0, C_1, C_2, \dots denote positive constants (possibly different), we use the notation $\|\cdot\|_p$ for the standard $L^p(\mathbb{R})$ -norm and $\|\cdot\|_*$ for the norm in the dual space X^* .

2 Some Preliminary Results

In this section, we prove some technical results about the space X and we show a version of (1.4) to X . First, in order to obtain good properties for X , we need the following lemma:

Lemma 2.1. *Suppose that (V_1) and (V_2) are satisfied. Then there exists $\kappa > 0$ such that for any $u \in X$,*

$$\frac{1}{2\pi} \left(\int_{\mathbb{R}^2} \frac{(u(x) - u(y))^2}{|x - y|^2} dx dy \right) + \int_{\mathbb{R}} V(x) u^2 dx \geq \kappa \|u\|_{1/2,2}^2. \quad (2.1)$$

Proof: Suppose that (2.1) is false. Then for each $n \in \mathbb{N}$ there exists $u_n \in X$ such that $\|u_n\|_{1/2,2}^2 = 1$ and

$$\frac{1}{2\pi} \left(\int_{\mathbb{R}^2} \frac{(u_n(x) - u_n(y))^2}{|x - y|^2} dx dy \right) + \int_{\mathbb{R}} V(x) u_n^2 dx < \frac{1}{n}.$$

Thus, by (V_2) it follows that

$$\lambda_1 \leq \frac{1}{\|u_n\|_2^2} \left(\frac{1}{2\pi} \left(\int_{\mathbb{R}^2} \frac{(u_n(x) - u_n(y))^2}{|x - y|^2} dx dy \right) + \int_{\mathbb{R}} V(x) u_n^2 dx \right) < \frac{1}{n \|u_n\|_2^2}.$$

This together with $\lambda_1 > 0$ imply $\|u_n\|_2 \rightarrow 0$ and $[u_n]_{1/2,2} \rightarrow 1$. Consequently, by using (V_1) we obtain the contradiction

$$o_n(1) = -B \|u_n\|_2^2 \leq \int_{\mathbb{R}} V(x) u_n^2 dx < \frac{1}{n} - \frac{1}{2\pi} [u_n]_{1/2,2} \rightarrow -\frac{1}{2\pi}.$$

This completes the proof. ■

Using (2.1) we can define the following inner product in X ,

$$\langle u, v \rangle := \frac{1}{2\pi} \left(\int_{\mathbb{R}^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^2} dx dy \right) + \int_{\mathbb{R}} V(x) uv dx, \quad u, v \in X, \quad (2.2)$$

to which corresponds the norm

$$\|u\| = \left\{ \frac{1}{2\pi} \left(\int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^2} dx dy \right) + \int_{\mathbb{R}} V(x) u^2 dx \right\}^{1/2}, \quad u \in X.$$

Moreover, the following facts hold:

- i)* X is a Hilbert space;
- ii)* X is continuously embedded into $H^{1/2}(\mathbb{R})$;
- iii)* For any $q \in [2, \infty)$, X is continuously embedded into $L^q(\mathbb{R})$ and

$$S_p := \inf_{\substack{u \in X \\ \|u\|_p = 1}} \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x) - u(y))^2}{|x - y|^2} dx dy + \int_{\mathbb{R}} V(x) u^2 dx \right)^{1/2} > 0. \quad (2.3)$$

Now, by adapting the arguments in [17], we prove a compactness result which will be used in this paper.

Lemma 2.2. *Suppose $(V_1) - (V_3)$ hold. Then X is compactly embedded into $L^q(\mathbb{R})$ for any $q \in [2, \infty)$.*

Proof: Let $(u_n) \subset X$ be a bounded sequence, up to a subsequence, we may assume that $u_n \rightharpoonup 0$ weakly in X . We must prove that, up to a subsequence,

$$u_n \rightarrow 0 \quad \text{strongly in } L^2(\mathbb{R}).$$

Let $\varphi \in C^\infty(\mathbb{R}, [0, 1])$ such that $\varphi \equiv 0$ in \overline{B}_R and $\varphi \equiv 1$ in $\mathbb{R} \setminus \overline{B}_{R+1}$. Then,

$$\begin{aligned} \|u_n\|_2 &= \|(1 - \varphi)u_n + \varphi u_n\|_2 \\ &\leq \|(1 - \varphi)u_n\|_2 + \|\varphi u_n\|_2 \\ &= \|(1 - \varphi)u_n\|_{L^2(B_{R+1})} + \|\varphi u_n\|_{L^2(\mathbb{R} \setminus \overline{B}_R)}. \end{aligned} \tag{2.4}$$

Since $H^{1/2}(B_{R+1})$ is compactly embedded into $L^2(B_{R+1})$, up to a subsequence,

$$\|(1 - \varphi)u_n\|_{L^2(B_{R+1})} \rightarrow 0. \tag{2.5}$$

Now, by the definition of $\nu(\mathbb{R} \setminus \overline{B}_R)$, it follows that

$$\|\varphi u_n\|_{L^2(\mathbb{R} \setminus \overline{B}_R)}^2 \leq \frac{\|\varphi u_n\|^2}{\nu(\mathbb{R} \setminus \overline{B}_R)} \leq \frac{C}{\nu(\mathbb{R} \setminus \overline{B}_R)},$$

which together with (V_3) , implies

$$\|\varphi u_n\|_{L^2(\mathbb{R} \setminus \overline{B}_R)} \rightarrow 0. \tag{2.6}$$

Finally, from (2.4), (2.5) and (2.6), we conclude the proof. \blacksquare

In the sequel we shall prove a version of (1.4) to the space X , this will be our main tool to prove Theorems 1.1 and 1.2. The ideas used in the proof are inspired in [6, 7, 8] and we present here for completeness of our work. For this, we need the following relation

$$\|(-\Delta)^{1/4} u\|_2 = (2\pi)^{-1/2} [u]_{1/2,2}, \quad \forall u \in H^{1/2}(\mathbb{R}), \tag{2.7}$$

for details see [5, Proposition 3.6].

Lemma 2.3. *If $0 < \alpha \leq 2\pi\kappa\omega$ and $u \in X$ with $\|u\| \leq 1$, then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1) dx \leq C. \tag{2.8}$$

Moreover, for any $\alpha > 0$ and $u \in X$, we have

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1) dx < \infty. \tag{2.9}$$

Proof: First, we observe that if a function $u \in X$ satisfies $\|u\| \leq 1$, set $v = (2\pi\kappa)^{1/2}u$, then $v \in H^{1/2}(\mathbb{R})$ and by using (2.1) and (2.7), we get

$$\|(-\Delta)^{1/4}v\|_2 = (2\pi)^{-1/2}[v]_{1/2,2} \leq \kappa^{1/2}\|u\|_{1/2,2} \leq \|u\| \leq 1.$$

Consequently,

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1) dx = \int_{\mathbb{R}} (e^{(\alpha/2\pi\kappa)v^2} - 1) dx \leq C_1\|v\|_2^2 \leq C,$$

where we have used (1.4). Thus, we obtain (2.8).

Now we prove the second part of the lemma. Indeed, given $u \in X$ and $\varepsilon > 0$ there exists $\varphi \in C_0^\infty(\mathbb{R})$ such that $\|u - \varphi\| < \varepsilon$. Thus, since

$$e^{\alpha u^2} - 1 \leq e^{\alpha(2(u-\varphi)^2 + 2\varphi^2)} - 1 \leq \frac{1}{2} \left(e^{4\alpha(u-\varphi)^2} - 1 \right) + \frac{1}{2} \left(e^{4\alpha\varphi^2} - 1 \right),$$

it follows that

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1) dx \leq \frac{1}{2} \int_{\mathbb{R}} (e^{4\alpha\|u-\varphi\|^2 \left(\frac{u-\varphi}{\|u-\varphi\|}\right)^2} - 1) dx + \frac{1}{2} \int_{\mathbb{R}} (e^{4\alpha\varphi^2} - 1) dx. \quad (2.10)$$

Choosing $\varepsilon > 0$ such that $4\alpha\varepsilon^2 < 2\pi\kappa\omega$, we have $4\alpha\|u - \varphi\|^2 < 2\pi\kappa\omega$. Then, from (2.8) and (2.10), we obtain that

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1) dx \leq \frac{C}{2} + \frac{1}{2} \int_{\text{supp}(\varphi)} (e^{4\alpha\varphi^2} - 1) dx < \infty.$$

This completes the proof. ■

The next result will be used to ensure the geometry of the functional I in Section 3.

Lemma 2.4. *If $v \in X$, $\alpha > 0$, $q > 2$ and $\|v\| \leq M$ with $\alpha M^2 < 2\pi\kappa\omega$, then there exists $C = C(\alpha, M, q) > 0$ such that*

$$\int_{\mathbb{R}} (e^{\alpha v^2} - 1) |v|^q dx \leq C \|v\|^q.$$

Proof: We consider $r > 1$ close to 1 such that $\alpha r M^2 < 2\pi\kappa\omega$ and $r'q \geq 2$, where $r' = r/(r-1)$. Using Hölder inequality, we have

$$\int_{\mathbb{R}} (e^{\alpha v^2} - 1) |v|^q dx \leq \left(\int_{\mathbb{R}} (e^{\alpha v^2} - 1)^r dx \right)^{1/r} \|v\|_{r'q}^q. \quad (2.11)$$

Note that given $\beta > r$ there exists $C = C(\beta) > 0$ such that for all $s \in \mathbb{R}$,

$$(e^{\alpha s^2} - 1)^r \leq C(e^{\alpha\beta s^2} - 1). \quad (2.12)$$

Hence, from (2.11) and (2.12), we get

$$\begin{aligned} \int_{\mathbb{R}} (e^{\alpha v^2} - 1) |v|^q dx &\leq C \left(\int_{\mathbb{R}} (e^{\alpha\beta v^2} - 1) dx \right)^{1/r} \|v\|_{r'q}^q \\ &\leq C \left(\int_{\mathbb{R}} (e^{\alpha\beta M^2 \left(\frac{v}{\|v\|}\right)^2} - 1) dx \right)^{1/r} \|v\|_{r'q}^q. \end{aligned}$$

By choosing $\beta > r$ close to r , in such way that $\alpha\beta M^2 < 2\pi\kappa\omega$, it follows from (2.8) and the continuous embedding $X \hookrightarrow L^{r'q}(\mathbb{R})$ that

$$\int_{\mathbb{R}} (e^{\alpha v^2} - 1) |v|^q dx \leq C \|v\|^q.$$

This completes the proof. ■

Now, in line with the Concentration-Compactness Principle due to P. -L. Lions [11] we show a refinement of (2.8). This result will be crucial to show that the functional I satisfies the Palais-Smale condition.

Lemma 2.5. *If (v_n) is a sequence in X with $\|v_n\| = 1$ for all $n \in \mathbb{N}$ and $v_n \rightharpoonup v$ in X , $0 < \|v\| < 1$, then for all $0 < t < 2\pi\kappa\omega(1 - \|v\|^2)^{-1}$, we have*

$$\sup_n \int_{\mathbb{R}} (e^{tv_n^2} - 1) dx < \infty.$$

Proof: Since $v_n \rightharpoonup v$ in X and $\|v_n\| = 1$, we conclude that

$$\|v_n - v\|^2 = 1 - 2\langle v_n, v \rangle + \|v\|^2 \rightarrow 1 - \|v\|^2 < \frac{2\pi\kappa\omega}{t}.$$

Thus, for $n \in \mathbb{N}$ enough large, we have $t\|v_n - v\|^2 < 2\pi\kappa\omega$. Now choosing $q > 1$ close to 1 and $\varepsilon > 0$ satisfying

$$qt(1 + \varepsilon^2)\|v_n - v\|^2 < 2\pi\kappa\omega.$$

Consequently, by (2.8), there exists $C > 0$ such that

$$\int_{\mathbb{R}} (e^{qt(1+\varepsilon^2)(v_n-v)^2} - 1) dx = \int_{\mathbb{R}} \left(e^{qt(1+\varepsilon^2)\|v_n-v\|^2 \left(\frac{v_n-v}{\|v_n-v\|} \right)^2} - 1 \right) dx \leq C. \quad (2.13)$$

Moreover, since

$$tv_n^2 \leq t(1 + \varepsilon^2)(v_n - v)^2 + t \left(1 + \frac{1}{\varepsilon^2} \right) v^2,$$

it follows by convexity of the exponential function with $q^{-1} + r^{-1} = 1$ that

$$e^{tv_n^2} - 1 \leq \frac{1}{q} (e^{qt(1+\varepsilon^2)(v_n-v)^2} - 1) + \frac{1}{r} (e^{rt(1+1/\varepsilon^2)v^2} - 1).$$

Therefore, by (2.9) and (2.13), we get

$$\int_{\mathbb{R}} (e^{tv_n^2} - 1) dx \leq \frac{1}{q} \int_{\mathbb{R}} (e^{qt(1+\varepsilon^2)(v_n-v)^2} - 1) dx + \int_{\mathbb{R}} (e^{rt(1+1/\varepsilon^2)v^2} - 1) dx \leq C,$$

and the result is proved. ■

3 The variational framework

As we mentioned in the introduction, the problem (1.1) has variational structure. In order to apply the critical point theory, we define the following functional $I : X \rightarrow \mathbb{R}$,

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} F(x, u) \, dx - (h, u).$$

Notice that, from (f_1) and (f_2) , for each $\alpha > \alpha_0$ and $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|F(x, s)| \leq \frac{(\lambda_1 - \varepsilon)}{2} s^2 + C_\varepsilon (e^{\alpha s^2} - 1), \quad \forall s \in \mathbb{R},$$

which together with the continuous embedding $X \hookrightarrow L^2(\mathbb{R})$ and (2.9) yields $F(x, u) \in L^1(\mathbb{R})$ for all $u \in X$. Consequently, I is well-defined and by standard arguments, $I \in C^1(X, \mathbb{R})$ with

$$(I'(u), v) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^2} \, dx \, dy + \int_{\mathbb{R}} V(x)uv \, dx - \int_{\mathbb{R}} f(x, u)v \, dx - (h, v),$$

for all $u, v \in X$. Hence, a critical point of I is a weak solution of (1.1) and reciprocally.

The geometric conditions of the mountain-pass theorem for the functional I are established by next lemmas.

Lemma 3.1. *Suppose that $(V_1) - (V_2)$ and $(f_1) - (f_2)$ hold. Then there exists $\delta_1 > 0$ such that for each $h \in X^*$ with $\|h\|_* < \delta_1$, there exists $\rho_h > 0$ such that*

$$I(u) > 0 \quad \text{if} \quad \|u\| = \rho_h.$$

Proof: From (f_1) and (f_2) , given $\varepsilon > 0$, there exists $C > 0$ such that, for all $\alpha > \alpha_0$ and $q > 2$,

$$|F(x, s)| \leq \frac{(\lambda_1 - \varepsilon)}{2} s^2 + C(e^{\alpha s^2} - 1)|s|^q, \quad \forall s \in \mathbb{R}. \quad (3.1)$$

Using (3.1) and (V_2) , we have

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \frac{(\lambda_1 - \varepsilon)}{2} \int_{\mathbb{R}} u^2 \, dx - C \int_{\mathbb{R}} (e^{\alpha u^2} - 1)|u|^q \, dx - \|h\|_* \|u\| \\ &\geq \frac{1}{2}\|u\|^2 - \frac{(\lambda_1 - \varepsilon)}{2\lambda_1} \|u\|^2 - C \int_{\mathbb{R}} (e^{\alpha u^2} - 1)|u|^q \, dx - \|h\|_* \|u\|. \end{aligned}$$

Then, for $u \in X$ such that $\alpha\|u\|^2 < 2\pi\kappa\omega$, using Lemma 2.4, we get

$$I(u) \geq \left(\frac{1}{2} - \frac{(\lambda_1 - \varepsilon)}{2\lambda_1} \right) \|u\|^2 - C\|u\|^q - \|h\|_* \|u\|.$$

Consequently,

$$I(u) \geq \|u\| \left[\left(\frac{1}{2} - \frac{(\lambda_1 - \varepsilon)}{2\lambda_1} \right) \|u\| - C\|u\|^{q-1} - \|h\|_* \right].$$

Since $\frac{1}{2} - \frac{(\lambda_1 - \varepsilon)}{2\lambda_1} > 0$, we may choose $\rho_h > 0$ such that

$$\left(\frac{1}{2} - \frac{(\lambda_1 - \varepsilon)}{2\lambda_1}\right) \rho_h - C\rho_h^{q-1} > 0.$$

Thus, for $\|h\|_*$ sufficiently small there exists ρ_h such that $I(u) > 0$ if $\|u\| = \rho_h$. ■

Lemma 3.2. *Assume that $(V_1) - (V_2)$ and $(f_2) - (f_3)$ hold. Then there exists $e \in X$ with $\|e\| > \rho_h$ such that*

$$I(e) < \inf_{\|u\|=\rho_h} I(u).$$

Proof: Let $u \in C_0^\infty(\mathbb{R}) \setminus \{0\}$, $u \geq 0$ with compact support $K = \text{supp}(u)$. By using (f_2) and (f_3) , there exist positive constants C_1 and C_2 such that

$$F(x, s) \geq C_1 s^\theta - C_2, \quad \forall (x, s) \in K \times [0, \infty).$$

Then, for $t > 0$, we get

$$I(tu) \leq \frac{t^2}{2} \|u\|^2 - C_1 t^\theta \int_K u^\theta dx + C_2 \int_K dx + t|(h, u)|.$$

Since $\theta > 2$, we have $I(tu) \rightarrow -\infty$ as $t \rightarrow \infty$. Setting $e = tu$ with t large enough, the proof is finished. ■

In order to find an appropriate ball to use minimization argument, we need the following results:

Lemma 3.3. *Suppose that $(V_1) - (V_2)$ and $(f_1) - (f_2)$ hold. Then if $h \neq 0$ there exist $\eta > 0$ and $v \in X \setminus \{0\}$ such that $I(tv) < 0$ for all $0 < t < \eta$. In particular,*

$$-\infty < c_0 \equiv \inf_{\|u\| \leq \eta} I(u) < 0.$$

Proof: For each $h \in X^*$, by applying the Riesz representation theorem in the space X , the problem

$$(-\Delta)^{1/2} u + V(x)u = h, \quad x \text{ in } \mathbb{R},$$

has a unique weak solution $v \in X$ so that

$$(h, v) = \|v\|^2 > 0.$$

Consequently, from (f_1) and (f_2) it follows that there exists $\eta > 0$ such that

$$\frac{d}{dt} I(tv) = t\|v\|^2 - \int_{\mathbb{R}} f(x, tv)v dx - (h, v) < 0,$$

for all $0 < t < \eta$. Using that $I(0) = 0$, it must hold that $I(tv) < 0$ for all $0 < t < \eta$ and the proof is completed. ■

4 Palais-Smale compactness condition

In this section we show that I satisfies the Palais-Smale condition for certain energy levels. We recall that the functional I satisfies the Palais-Smale condition at level c , denoted by $(PS)_c$, if for any sequence (u_n) in X such that

$$I(u_n) \rightarrow c \quad \text{and} \quad I'(u_n) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty, \quad (4.1)$$

has a strongly convergent subsequence in X .

Initially, we prove the following lemma:

Lemma 4.1. *Suppose that $(V_1) - (V_3)$ and $(f_1) - (f_4)$ are satisfied. Let $(u_n) \subset X$ be an arbitrary Palais-Smale sequence of I at level c . Then there exists a subsequence of (u_n) (still denoted by (u_n)) and $u \in X$ such that*

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } X, \\ f(x, u_n) \rightarrow f(x, u) & \text{in } L^1_{loc}(\mathbb{R}), \\ F(x, u_n) \rightarrow F(x, u) & \text{in } L^1(\mathbb{R}). \end{cases}$$

Proof: Note that by (f_3) ,

$$\begin{aligned} I(u_n) - \frac{1}{\theta}(I'(u_n), u_n) &= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|^2 + \int_{\mathbb{R}} \left(\frac{1}{\theta} f(x, u_n) u_n - F(x, u_n)\right) dx + \left(\frac{1}{\theta} - 1\right) (h, u_n) \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|^2 + \left(\frac{1}{\theta} - 1\right) (h, u_n). \end{aligned} \quad (4.2)$$

By using (4.1), we obtain

$$I(u_n) - \frac{1}{\theta}(I'(u_n), u_n) \leq C + \|u_n\|.$$

This together with (4.2) leads to $\|u_n\| \leq C$. Hence, since that X is a Hilbert space, up to subsequence, we can assume that there exists $u \in X$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } X, \\ u_n \rightarrow u & \text{in } L^q(\mathbb{R}), \quad \forall q \in [2, \infty), \\ u_n(x) \rightarrow u(x) & \text{almost everywhere in } \mathbb{R}. \end{cases}$$

Now, from (4.1) and $\|u_n\| \leq C$, there exists $C_1 > 0$ such that

$$\int_{\mathbb{R}} |f(x, u_n) u_n| \leq C_1.$$

Consequently, thanks to Lemma 2.1 in [4], we get

$$f(x, u_n) \rightarrow f(x, u) \quad \text{in } L^1_{loc}(\mathbb{R}). \quad (4.3)$$

Next, similar to N. Lam and G. Lu [10], we shall prove the last convergence of the lemma. Firstly, note that by using (f_3) and (f_4) , for each $R > 0$ there exists $C_0 > 0$ such that

$$F(x, u_n) \leq C_0 |f(x, u_n)|.$$

This together with (4.3) and the Generalized Lebesgue's Dominated Convergence Theorem, imply

$$F(x, u_n) \rightarrow F(x, u) \text{ in } L^1(B_R), \quad \forall R > 0.$$

Now, to conclude the last convergence of the lemma, it is sufficient to prove that given $\delta > 0$, there exists $R > 0$ such that

$$\int_{B_R^c} F(x, u_n) dx \leq \delta \text{ and } \int_{B_R^c} F(x, u) dx \leq \delta.$$

In order to prove it, we notice that by using (f_1) , (f_3) and (f_4) , there exist $C_1, C_2 > 0$ such that

$$|F(x, s)| \leq C_1 |s|^2 + C_2 |f(x, s)|, \quad \forall (x, s) \in \mathbb{R} \times \mathbb{R},$$

Thus, for each $A > 0$, we obtain that

$$\begin{aligned} \int_{\substack{|x| > R \\ |u_n| > A}} F(x, u_n) dx &\leq C_1 \int_{\substack{|x| > R \\ |u_n| > A}} |u_n|^2 dx + C_2 \int_{\substack{|x| > R \\ |u_n| > A}} |f(x, u_n)| dx \\ &\leq \frac{C_1}{A} \int_{\substack{|x| > R \\ |u_n| > A}} |u_n|^3 dx + \frac{C_2}{A} \int_{\mathbb{R}} |f(x, u_n) u_n| dx \\ &\leq \frac{C_1}{A} \|u_n\|^3 + \frac{C_2}{A} \int_{\mathbb{R}} |f(x, u_n) u_n| dx. \end{aligned}$$

Since $\|u_n\|$ and $\int_{\mathbb{R}} |f(x, u_n) u_n| dx$ are bounded, given $\delta > 0$ we may choose A such that

$$\frac{C_1}{A} \|u_n\|^3 < \delta/3 \quad \text{and} \quad \frac{C_2}{A} \int_{\mathbb{R}} |f(x, u_n) u_n| dx < \delta/3.$$

Thus,

$$\int_{\substack{|x| > R \\ |u_n| > A}} F(x, u_n) dx \leq 2\delta/3. \tag{4.4}$$

Moreover, note that with such A , by (f_1) and (f_2) , we have that

$$F(x, s) \leq C(\alpha_0, A) |s|^2, \quad \forall (x, s) \in \mathbb{R} \times [-A, A].$$

So, we get

$$\begin{aligned} \int_{\substack{|x| > R \\ |u_n| \leq A}} F(x, u_n) dx &\leq C(\alpha_0, A) \int_{\substack{|x| > R \\ |u_n| \leq A}} |u_n|^2 dx \\ &\leq 2C(\alpha_0, A) \int_{\substack{|x| > R \\ |u_n| \leq A}} |u_n - u|^2 dx + 2C(\alpha_0, A) \int_{\substack{|x| > R \\ |u_n| \leq A}} |u|^2 dx. \end{aligned}$$

Hence, using Lemma 2.2, given $\delta > 0$ we may choose $R > 0$ such that

$$\int_{\substack{|x| > R \\ |u_n| \leq A}} F(x, u_n) \, dx \leq \delta/3. \quad (4.5)$$

From (4.4) and (4.5), we have that given $\delta > 0$ there exists $R > 0$ such that

$$\int_{|x| > R} F(x, u_n) \, dx \leq \delta.$$

Similarly,

$$\int_{|x| > R} F(x, u) \, dx \leq \delta.$$

Combining all the above estimates and since that $\delta > 0$ is arbitrary, we have

$$\int_{\mathbb{R}} F(x, u_n) \, dx \rightarrow \int_{\mathbb{R}} F(x, u) \, dx,$$

which completes the proof. ■

Now, we shall prove main results this Section.

Proposition 4.2. *Under the hypotheses $(V_1) - (V_3)$ and $(f_1) - (f_4)$, the functional I satisfies $(PS)_c$ for any $c < \pi\kappa\omega/\alpha_0$.*

Proof: Let $(u_n) \subset X$ be an arbitrary Palais-Smale sequence of I at level c . By Lemma 4.1, up to a subsequence, $u_n \rightharpoonup u$ weakly in X .

We shall show that, up to a subsequence, $u_n \rightarrow u$ strongly in X . For this, we have two cases to consider:

Case 1: $u = 0$.

In this case, again by Lemma 4.1, we have

$$\int_{\mathbb{R}} F(x, u_n) \rightarrow 0 \quad \text{and} \quad (h, u_n) \rightarrow 0.$$

Since

$$I(u_n) = \frac{1}{2} \|u_n\|^2 - \int_{\mathbb{R}} F(x, u_n) - (h, u_n) = c + o_n(1),$$

we get

$$\lim_{n \rightarrow \infty} \|u_n\|^2 = 2c.$$

Hence, we can infer that for n large there exist $r_1 > 1$ sufficiently close to 1, $\alpha > \alpha_0$ close to α_0 and $\tilde{r}_1 > r_1$ sufficiently close to r_1 such that $\tilde{r}_1 \alpha \|u_n\|^2 < 2\pi\kappa\omega$. Thus, by (2.12) and (2.8), we have

$$\int_{\mathbb{R}} (e^{\alpha u_n^2} - 1)^{r_1} \, dx \leq C \int_{\mathbb{R}} (e^{\tilde{r}_1 \alpha \|u_n\|^2 \left(\frac{u_n}{\|u_n\|}\right)^2} - 1) \, dx \leq C. \quad (4.6)$$

Consequently,

$$\int_{\mathbb{R}} f(x, u_n) u_n \, dx \rightarrow 0.$$

In fact, since $f(x, s)$ satisfies (f_1) and (f_2) , for $\alpha > \alpha_0$ and $\varepsilon > 0$, there exists $C_1 > 0$ such that

$$|f(x, s)| \leq (\lambda_1 - \varepsilon)|s| + C_1(e^{\alpha s^2} - 1), \quad \forall s \in \mathbb{R}.$$

Now, letting $r_1 > 1$ sufficiently close to 1 such that $r_2 \geq 2$, where $1/r_1 + 1/r_2 = 1$, we obtain by Hölder inequality that

$$\left| \int_{\mathbb{R}} f(x, u_n) u_n \, dx \right| \leq C \int_{\mathbb{R}} |u_n|^2 \, dx + C \left(\int_{\mathbb{R}} (e^{\alpha u_n^2} - 1)^{r_1} \, dx \right)^{1/r_1} \left(\int_{\mathbb{R}} |u_n|^{r_2} \, dx \right)^{1/r_2} \rightarrow 0,$$

where we have used (4.6) and Lemma 2.2.

Therefore, since $(I'(u_n), u_n) = o_n(1)$, we conclude that, up to a subsequence, $u_n \rightarrow 0$ strongly in X .

Case 2: $u \neq 0$.

In this case, we define

$$v_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad v = \lim_{n \rightarrow \infty} \frac{u}{\|u_n\|}.$$

It follows that $v_n \rightharpoonup v$ in X , $\|v_n\| = 1$ and $\|v\| \leq 1$. Thus, if $\|v\| = 1$, we conclude the proof. Now, if $\|v\| < 1$, we claim that there exist $r_1 > 1$ sufficiently close to 1, $\alpha > \alpha_0$ close to α_0 and $\beta > 0$ such that

$$r_1 \alpha \|u_n\|^2 \leq \beta < 2\pi\kappa\omega(1 - \|v\|^2)^{-1} \quad (4.7)$$

for $n \in \mathbb{N}$ large.

Indeed, since $I(u_n) = c + o_n(1)$, it follows that

$$\frac{1}{2} \lim_{n \rightarrow \infty} \|u_n\|^2 = c + \int_{\mathbb{R}} F(x, u) \, dx + (h, u). \quad (4.8)$$

Setting

$$A = \left(c + \int_{\mathbb{R}} F(x, u) \, dx + (h, u) \right) (1 - \|v\|^2),$$

from (4.8) and by definition of v , we obtain

$$A = c - I(u),$$

which together with (4.8) imply

$$\frac{1}{2} \lim_{n \rightarrow \infty} \|u_n\|^2 = \frac{A}{1 - \|v\|^2} = \frac{c - I(u)}{1 - \|v\|^2} < \frac{\pi\kappa\omega}{\alpha_0(1 - \|v\|^2)}.$$

Consequently, (4.7) holds. Again by (2.12) and Lemma 2.5, we get

$$\int_{\mathbb{R}} (e^{\alpha u_n^2} - 1)^{r_1} dx \leq C.$$

By Hölder inequality and similar computations done above we have that

$$\int_{\mathbb{R}} f(x, u_n)(u_n - u) dx \rightarrow 0.$$

This convergence together with the fact that $(I'(u_n), (u_n - u)) = o_n(1)$ imply that

$$\|u_n\|^2 = (u_n, u) + o_n(1).$$

Since $u_n \rightharpoonup u$ weakly in X , we obtain $u_n \rightarrow u$ strongly in X and the proof is finished. ■

5 Minimax Level

In this section, we verify that the minimax level associated with the Mountain Pass Theorem is in the interval where the Proposition 4.2 can be applied. To show this result the idea is to find a nonnegative function $u_p \in X$ which attains S_p . And then we show the main result of the section, providing the estimate for $\max_{t \geq 0} I(tu_p)$.

Lemma 5.1. *Suppose that $(V_1) - (V_3)$ hold. Then S_p is attained by a non-negative function $u_p \in X$.*

Proof: Let (u_n) be a minimizing sequence of non-negative functions (if necessary, replace u_n by $|u_n|$) for S_p in X , that is,

$$\|u_n\|_p = 1 \quad \text{and} \quad \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u_n(x) - u_n(y))^2}{|x - y|^2} dx dy + \int_{\mathbb{R}} V(x) u_n^2 dx \right)^{1/2} \rightarrow S_p.$$

Then, (u_n) is bounded in X . Since X is Hilbert and X is compactly embedded into $L^p(\mathbb{R})$, up to a subsequence, we may assume

$$\begin{aligned} u_n &\rightharpoonup u_p \quad \text{weakly in } X, \\ u_n &\rightarrow u_p \quad \text{strongly in } L^p(\mathbb{R}), \\ u_n(x) &\rightarrow u_p(x) \quad \text{almost everywhere in } \mathbb{R}. \end{aligned}$$

Consequently, we have

$$\|u_p\| = 1, u_p(x) \geq 0 \quad \text{and} \quad \|u_p\| \leq \liminf_{n \rightarrow +\infty} \|u_n\| = S_p.$$

Thus, $S_p = \|u_p\|$. This completes the proof. ■

Now we prove the main result of this section.

Lemma 5.2. *Suppose that $(V_1) - (V_3)$ and (f_5) are satisfied, if $\|h\|_*$ is sufficiently small then*

$$\max_{t \geq 0} I(tu_p) < \frac{\pi\kappa\omega}{\alpha_0}.$$

Proof: Let $\Psi : [0, +\infty) \rightarrow \mathbb{R}$ given by

$$\Psi(t) = \frac{t^2}{2} \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u_p(x) - u_p(y))^2}{|x - y|^2} dx dy + \int_{\mathbb{R}} V(x) u_p^2 dx \right) - \int_{\mathbb{R}} F(x, tu_p) dx.$$

By Lemma 5.1 and (f_5) , we get

$$\Psi(t) \leq \frac{t^2}{2} S_p^2 - \frac{C_p}{p} t^p \leq \max_{t \geq 0} \left[\frac{t^2}{2} S_p^2 - \frac{C_p}{p} t^p \right] = \frac{(p-2)}{2p} \frac{S_p^{2p/(p-2)}}{C_p^{2/(p-2)}} < \frac{\pi\kappa\omega}{\alpha_0}. \quad (5.1)$$

To conclude, note that $|(h, u_p)| \leq \|h\|_* \|u_p\|$. Thus taking $\|h\|_*$ sufficiently small and using (5.1) the result follows. \blacksquare

6 Proofs of Theorem 1.1 and 1.2

By Lemmas 3.1, 3.2 the functional I satisfies the geometric properties of the Mountain Pass Theorem. As a consequence, the minimax level

$$c_m = \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)) > 0,$$

where $\Gamma = \{g \in C([0,1], X) : g(0) = 0, g(1) = e\}$.

By Lemma 5.2 and Proposition 4.2, the functional I satisfies the $(PS)_{c_m}$ condition. Therefore, by the mountain-pass Theorem the functional I has a critical point u_m at the minimax level c_m .

Moreover, if $h \in X^*$ with $h \neq 0$, we can find a second solution. To this, we consider ρ_h like in Lemma 3.1. Observe that \overline{B}_{ρ_h} is a complete metric space with the metric induced by the norm of X and convex, and the functional I is of class C^1 and bounded below on \overline{B}_{ρ_h} . Thus, by Ekeland variational principle there exists a sequence (u_n) in \overline{B}_{ρ_h} such that

$$I(u_n) \rightarrow c_0 = \inf_{\|u\| \leq \rho_h} I(u) < 0 \quad \text{and} \quad \|I'(u_n)\|_* \rightarrow 0.$$

Hence, by Proposition 4.2 the functional I satisfies the $(PS)_{c_0}$ condition. Consequently, there exists $u_0 \in X$ such that $I'(u_0) = 0$ and $I(u_0) = c_0$, that is, u_0 is a weak solution of (1.1) at level c_0 .

Thus it is completed the proof of the results.

References

- [1] A. Ambrosetti; P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

- [2] T. Bartsch; Z.-Q. Wang, *Existence and multiplicity results for some superlinear elliptic problems on \mathbb{R}^N* , Comm. Part. Diff. Eq. **20** (1995), 1725–1741.
- [3] M. Cheng, *Bound state for the fractional Schrödinger equation with unbounded potential*, J. Math. Phys. **53** (2012), 043507.
- [4] D. G. de Figueiredo; O. H. Miyagaki; B. Ruf, *Elliptic equations in \mathbb{R}^2 with nonlinearities in the critical growth range*, Calc. Var. Partial Differential Equations **3** (1995), 139–153.
- [5] E. Di Nezza; G. Palatucci; E. Valdinoci, *Hitchhiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.
- [6] J. M. do Ó; M. de Souza, *On a class of singular Trudinger-Moser inequalities*, Mathematische Nachrichten **284** (2011), 1754–1776.
- [7] J. M. do Ó; E. Medeiros; U. B. Severo, *A nonhomogeneous elliptic problem involving critical growth in dimension two*, J. Math. Anal. Appl. **345** (2008), 286–304.
- [8] A. Iannizzotto; M. Squassina, *1/2-laplacian problems with exponential nonlinearity*, J. Math. Anal. Appl. **414** (2014), 372–385.
- [9] H. Kozono; T. Sato; H. Wadade, *Upper bound of the best constant of a Trudinger-Moser inequality and its application to a Gagliardo-Nirenberg inequality*, Indiana Univ. Math. J. **55** (2006), 1951–1974.
- [10] N. Lam; G. Lu, *Existence and multiplicity of solution to equations of N -Laplacian type with critical exponential growth in \mathbb{R}^N* , J. Funct. Anal. **262** (2012), 1132–1165.
- [11] P. -L. Lions, *The concentration-compactness principle in the calculus of variations, Part I*, Rev. Mat. Iberoamericana **1** (1985), 145–201.
- [12] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092.
- [13] T. Ozawa, *On critical cases of Sobolev’s inequalities*, J. Funct. Anal. **127** (1995), 259–269.
- [14] P. H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), 270–291.
- [15] R. Servadei; E. Valdinoci, *Mountain pass solution for non-local elliptic operators*, J. Math. Anal. Appl. **389** (2012), 887–898.
- [16] S. Secchi, *Ground state solutions for nonlinear fractional Schrödinger equations in \mathbb{R}^N* , J. of Math. Phys. **54** (2013), 031501.
- [17] B. Sirakov, *Existence and multiplicity of solutions of semi-linear elliptic equations in \mathbb{R}^N* , Cal. Var. Partial Differential Equations **11** (2000), 119–142.

- [18] N. S. Trudinger, *On the imbedding into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–484.
- [19] Y. Yang, *Existence of positive solutions to quasi-linear elliptic equations with exponential growth in the whole Euclidean space*, J. Funct. Anal. **262** (2012), 1679–1704.
- [20] M. Weinstein, *Solitary waves of nonlinear dispersive evolution equations with critical power nonlinearities*, Jour. Diff. Equa. **69** (1987) 192–203.